DUAL-MODE DISTRIBUTED MODEL PREDICTIVE CONTROL OF A QUADRUPLE-TANK SYSTEM

Alexandra Grancharova¹, Tor A. Johansen², Sorin Olaru³

1 Department of Industrial Automation
University of Chemical Technology and Metallurgy
Kliment Ohridski 8, Sofia 1756, Bulgaria
E-mail: alexandra.grancharova@abv.bg
2 Center for Autonomous Marine Operations and Systems
Department of Engineering Cybernetics, Norwegian University of Science and Technology
7491 Trondheim, Norway
E-mail: Tor.Arne.Johansen@itk.ntnu.no
3 Laboratory of Signals and Systems, CentraleSupélec-CNRS-Université Paris-Sud
Université Paris-Saclay, 3 Rue Joliot-Curie, 91192 Gif-sur-Yvette Cedex, France
E-mail: Sorin.Olaru@centralesupelec.fr

ABSTRACT

In this paper, a dual-mode distributed Model Predictive Control (MPC) approach is proposed in order to reduce the on-line computational complexity of the distributed optimal control of nonlinear interconnected systems. It consists in using a nonlinear distributed MPC approach when the state variables of the overall system are far from the origin and applying a linear distributed MPC method in a neighborhood of the origin. The nonlinear distributed approach is based on first-principles (nonlinear) models of the interconnected systems dynamics. It includes a sequential linearization of these models and finding distributedly a suboptimal solution of the resulting quadratic programming problem. In order to apply the linear distributed MPC method, it is necessary first to obtain a linearized model of the overall nonlinear system in a neighborhood of the origin. The benefit of the suggested dual-mode distributed MPC approach is the reduced complexity of the on-line computations in comparison to the entirely nonlinear approach when the current overall system state is in a neighborhood of the origin. The proposed method is illustrated with simulations on the model of a quadruple-tank system.

Keywords: distributed Model Predictive Control, nonlinear interconnected systems, quadruple-tank system.

INTRODUCTION

Model Predictive Control (MPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints [1-3]. The centralized solution of MPC problems for large-scale systems may be impractical due to the topology of the plant and data communication, and the large number of decision variables. Recently, several methods for distributed/decentralized MPC have been developed [4, 5]. In [2, 6 - 8], approaches for distributed/decentralized MPC for systems consisting of linear interconnected subsystems have been developed. In [2], a distributed optimization algorithm based on accelerated gradient methods using dual decomposition is proposed and its performance is evaluated in distributed MPC.

Also, approaches to distributed MPC for systems composed of several nonlinear subsystems have been proposed. Some of them assume the subsystems are
coupled through their dynamics [1, 9, 10], while others suppose that the dynamics of the nonlinear subsystems are not interconnected, but the cost function and/or constraints couple the dynamical behavior of the subsystems (e.g. [11]). The approach in [1] includes a sequential linearization of the first-principles (nonlinear) models of the interconnected systems dynamics and finding distributedly a suboptimal solution of the resulting quadratic programming problem.

In this paper, a dual-mode distributed MPC approach is proposed in order to reduce the on-line computational complexity of the distributed optimal control of nonlinear interconnected systems. It consists in using the distributed quasi-NMPC approach [1] when the state variables are far from the origin and the linear distributed MPC method [2] in a neighborhood of the origin. The proposed method is illustrated with simulations on the model of a quadruple-tank system.

FORMULATION OF MPC PROBLEM FOR NONLINEAR INTERCONNECTED SYSTEMS

Consider a system composed by the interconnection of \( M \) subsystems (Fig. 1.) with overall state and overall control input:

\[
\begin{align*}
    x(t) &= [x_1(t), x_2(t), \ldots, x_M(t)] \in \mathbb{R}^n, \quad n = \sum_{i=1}^M n_i \\
    u(t) &= [u_1(t), u_2(t), \ldots, u_M(t)] \in \mathbb{R}^m, \quad m = \sum_{i=1}^M m_i
\end{align*}
\]  

where \( x_i(t) \in \mathbb{R}^{n_i} \) and \( u_i(t) \in \mathbb{R}^{m_i} \) are the state and the control input, related to the \( i \)-th subsystem.

Let the dynamics of the subsystems be described by the nonlinear discrete-time models:

\[
    x_i(t+1) = f_i(x_i(t), u_i(t)), \quad i = 1, 2, \ldots, M
\]  

where \( f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i} \) is a nonlinear function. The constraints imposed on the subsystems are:

\[
    x_i(t) \in X_i, \quad u_i(t) \in U_i, \quad i = 1, 2, \ldots, M
\]  

where \( X_i \) and \( U_i \) are the admissible sets, and the following assumptions are made:

A1. The functions \( f_i \), \( i = 1, \ldots, M \) are continuously differentiable with \( f_i(0,0) = 0 \).

A2. The admissible sets \( X_i \) and \( U_i \) are bounded polyhedral sets, i.e. they are defined by:

\[
    X_i = \{ x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \leq d_i^x \}
\]  

\[
    U_i = \{ u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \leq d_i^u \}
\]  

and they include the origin in their interior. Here, \( C_i^x \in \mathbb{R}^{n_i \times n_{c,x}} \), \( C_i^u \in \mathbb{R}^{n_i \times n_{c,u}} \), \( d_i^x \in \mathbb{R}^{n_i} \), \( d_i^u \in \mathbb{R}^{n_i} \) and \( n_{c,x} \) and \( n_{c,u} \) are the number of constraints imposed on \( x_i \) and \( u_i \), respectively.

It can be seen from (4)-(6) that the constraints imposed on the subsystems are not coupled. In order to consider coupled constraints, the approach described in the next section should be slightly modified.

Fig. 1. System composed by the interconnection of \( M \) subsystems.
It is supposed that a full measurement $\mathbf{x} = [x_1, x_2, \ldots, x_M]$ of the overall state is available at the current time $t$. The optimal regulation problem is considered where the goal is to steer the overall state of the system (3) to the origin. For the current overall state $\mathbf{x}$, the regulation NMPC solves the optimization problem:

**Problem P1 (Centralized NMPC):**

$$V^\text{opt}(\mathbf{x}) = \min_U J(U, \mathbf{x})$$

subject to $x_{i,t} = \mathbf{x}$ and:

$$x_{i,t+k} \in \mathcal{X}_i, \quad i = 1, \ldots, M, \quad k = 1, \ldots, N$$

$$u_{i,t+k} \in \mathcal{U}_i, \quad i = 1, \ldots, M, \quad k = 0, 1, \ldots, N - 1$$

$$x_{i,t+k+1} = f_i(x_{i,t+k}, u_{i,t+k})$$

$$i = 1, \ldots, M, \quad k = 0, 1, \ldots, N - 1$$

$$x_{t+k+1} = [x_{1,t+k+1}, x_{2,t+k+1}, \ldots, x_{M,t+k+1}]$$

$$k = 0, 1, \ldots, N$$

$$u_{t+k} = [u_{1,t+k}, u_{2,t+k}, \ldots, u_{M,t+k}])$$

$$k = 0, 1, \ldots, N - 1$$

with $U = [u_t, u_{t+1}, \ldots, u_{t+N-1}]$ and the cost function given by:

$$J(U, \mathbf{x}) = \sum_{k=0}^{N} \sum_{i=1}^{M} l_i(x_{i,t+k}, u_{i,t+k})$$

where $l_i(x_{i,t+k}, u_{i,t+k}) = \|x_{i,t+k}\|_{Q_i}^2 + \|u_{i,t+k}\|_{R_i}^2$ is the stage cost for the $i$-th subsystem with symmetric weighting matrices $Q_i$, $R_i > 0$, and $N$ is a finite horizon.

**DUAL-MODE DISTRIBUTED MPC APPROACH**

Here, a dual-mode distributed MPC approach for interconnected nonlinear systems is suggested. It combines the distributed quasi-NMPC approach [1] and the distributed linear MPC approach [2].

**Distributed quasi-nonlinear MPC by sequential linearization and accelerated gradient method**

In [1], the dynamics of the subsystems (3) are locally approximated by linear models.

Let at time $t$, $U_i^0 = [u_{i,t+1}^0, u_{i,t+2}^0, \ldots, u_{i,t+N-1}^0]$ and $X_i^0 = [x_{i,t+1}^0, x_{i,t+2}^0, \ldots, x_{i,t+N-1}^0]$ be given trajectories of the control input and the state of the $i$-th subsystem for the prediction horizon $N$. Taylor series expansion of the right-hand side of the model (3) about the point $(U_i^0, X_i^0)$ leads to the locally linear model:

$$x_{i,t+k+1} = \sum_{j=1}^{M} (A_{i,j,k} x_{j,t+k} + B_{i,j,k} u_{j,t+k}) + g_{i,t+k}$$

$$k = 0, 1, \ldots, N - 1, \quad i = 1, \ldots, M$$

where the matrices $A_{i,j,k}$, $B_{i,j,k}$ and the vector $g_{i,t+k}$ are computed as:

$$A_{i,j,k} = \nabla_{x_i} f_i(x_{i,t+k}, u_{i,t+k})$$

$$B_{i,j,k} = \nabla_{u_i} f_i(x_{i,t+k}, u_{i,t+k})$$

$$g_{i,t+k} = -\sum_{j=1}^{M} (A_{i,j,k} x_{j,t+k} + B_{i,j,k} u_{j,t+k}) + f_i(x_{i,t+k}, u_{i,t+k})$$

$$k = 0, 1, \ldots, N - 1, \quad i = 1, \ldots, M$$

In (15), $U_{t+k} = [u_{1,t+k}, u_{2,t+k}, \ldots, u_{M,t+k}]$ and $X_{t+k} = [x_{1,t+k}, x_{2,t+k}, \ldots, x_{M,t+k}]$.

It can be observed that (14)-(15) is a linear time-varying approximation of the model (3).

As in [8], the following tightened constraint sets are introduced:

$$(1 - \delta) \mathcal{X}_i = \{ x_i \in \mathbb{R}^n \mid C_i x_i \leq (1 - \delta) d_i \}$$

$$(1 - \delta) \mathcal{U}_i = \{ u_i \in \mathbb{R}^m \mid C_i u_i \leq (1 - \delta) d_i \}$$

where $\delta \in (0, 1)$ is the amount of relative constraint tightening. Then, for the locally linear dynamics (14)-(15) with initial state $\mathbf{x} = [x_{1,t+1}^0, x_{2,t+1}^0, \ldots, x_{M,t+1}^0]$, the linearized MPC problem is formulated:

**Problem P2 (Centralized linearized MPC):**

$$V^*(\mathbf{x}) = \min_U J(U, \mathbf{x})$$

subject to $x_{i,t} = \mathbf{x}$, constraints (14)-(15) and:
\( x_{i,t+k} \in (1-\delta)X_i \), \( i = 1, \ldots, M \), \( k = 1, \ldots, N \)  
(19)

\( u_{i,t+k} \in (1-\delta)U_i \), \( i = 1, \ldots, M \), \( k = 0, 1, \ldots, N - 1 \)  
(20)

where the cost function \( J(U, \bar{x}) \) is defined by (13).

As it is shown in [2], by stacking all decision variables (the control input trajectory and the state trajectory along the horizon) into one vector \( Y \in \mathbb{R}^{n_y} \) with dimension \( n_y = \sum_{i=1}^{M} N(n_i + m_i) \):

\[
Y = [x_{1,t+1}, u_{1,t}, x_{1,t+2}, u_{1,t+1}, \ldots, x_{1,t+N}, u_{1,t+N-1}, \\
\vdots \]

\[
x_{M,t+1}, u_{M,t}, x_{M,t+2}, u_{M,t+1}, \ldots, x_{M,t+N}, u_{M,t+N-1}] 
\]  
(21)

the optimization problem \( P2 \) can be written as a Quadratic Programming (QP) problem:

**Problem P3 (QP problem):**

\[
V^*(\bar{x}) = \min \frac{1}{2} Y^T \hat{H} Y 
\]  
(22)

subject to:

\[
\bar{A} Y = \bar{B} \bar{x} - \bar{G} 
\]  
(23)

\[
\bar{C} Y \leq (1-\delta)\bar{d} 
\]  
(24)

Here, \( \hat{H} \in \mathbb{R}^{n_y \times n_y} \), \( \bar{A} \in \mathbb{R}^{n_x \times n_y} \), \( \bar{B} \in \mathbb{R}^{n_u \times n_y} \), \( \bar{C} \in \mathbb{R}^{n_{C} \times n_y} \), \( \bar{G} \in \mathbb{R}^{n_{G}} \), \( \bar{C} \in \mathbb{R}^{n_{G} \times n_y} \), \( \bar{A} \in \mathbb{R}^{n_{A} \times n_{A}} \) (\( n_x = \sum_{i=1}^{M} (N-1)n_i \), \( n_u = \sum_{i=1}^{M} n_i \), \( n_{C} = \sum_{i=1}^{M} N(n_{c,i} + n_{d,i}) \)). In (22)-(24) we have:

\[
\hat{H} = \text{diag}\{\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_M\} 
\]  
(25)

\[
\bar{A} = [\bar{A}_1 | \bar{A}_2 | \ldots | \bar{A}_M]^T 
\]  
(26)

\[
\bar{B} = [\bar{B}_1 | \bar{B}_2 | \ldots | \bar{B}_M]^T 
\]  
(26)

\[
\bar{G} = [\bar{G}_1 | \bar{G}_2 | \ldots | \bar{G}_M]^T 
\]  
(26)

\[
\bar{C} = \text{diag}\{\bar{C}_1 | \bar{C}_2 | \ldots | \bar{C}_M\}, \bar{d} = \text{diag}\{\bar{d}_1 | \bar{d}_2 | \ldots | \bar{d}_M\} 
\]  
(27)

For the \( i \)-th subsystem the matrix \( \hat{H}_i \in \mathbb{R}^{n_y \times n_y} \) (\( n_{y_i} = N(n_i + m_i) \)) is defined as:

\[
\hat{H}_i = \text{diag}\{W_{i,1}, W_{i,2}, \ldots, W_{i,N}\}, W_i = \begin{bmatrix} Q_i & 0 \\ 0 & R_i \end{bmatrix} \]  
(28)

and the matrices \( \bar{A}_i \in \mathbb{R}^{n_x \times n_y} \), \( \bar{B}_i \in \mathbb{R}^{n_u \times n_y} \), \( \bar{C}_i \in \mathbb{R}^{n_{C_i \times n_y}} \) and the vector \( \bar{d}_i \in \mathbb{R}^{n_{C_i}} \) (\( n_{C_i} = (N-1)n_i \))

\[
n_{C_{ij}} = N(n_{c,ij} + n_{c,ij}) \), \( i = 1, \ldots, M \) are:

\[
\bar{A}_i = \begin{bmatrix} \bar{A}_{i,1,1} & \bar{A}_{i,1,2} & \ldots & \bar{A}_{i,1,M} \\ \bar{A}_{i,2,1} & \bar{A}_{i,2,2} & \ldots & \bar{A}_{i,2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{i,N-1,1} & \bar{A}_{i,N-1,2} & \ldots & \bar{A}_{i,N-1,M} \\ -A_{i,1,i} - A_{i,2,i} - \cdots - A_{i,M,i} \end{bmatrix} 
\]  
(29)

\[
\bar{B}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} 
\]  
(30)

\[
\bar{C}_i = \text{diag}\{C_{i,x}^{x,u}, C_{i,x}^{x,u}, \ldots, C_{i,y}^{x,u}\} 
\]  
(30)

\[
\bar{d}_i = \text{diag}\{d_{i,x}^{x,u}, d_{i,x}^{x,u}, \ldots, d_{i,y}^{x,u}\}, d_{i,y}^{x,u} = [d_{i,x}^{x,u}, d_{i,y}^{x,u}]^T \]  
(31)

In (29), \( \bar{A}_{k,i,j}, k = 1, 2, \ldots, N - 1, j = 1, 2, \ldots, M \) depend on the matrices \( A_{i,t+k} \) and \( B_{i,t+k} \) of the linear model (14)-(15) and \( A_{i,t,j}, A_{i,t,j}, \ldots, A_{i,M,j} \) are the matrices \( A_{i,t+k} \) for \( k = 0 \), \( j = 1, 2, \ldots, M \).

The linearized MPC problem \( P2 \) can be solved distributedly by applying the dual accelerated gradient algorithm in [2]. The distribution is enabled by solving the dual problem to problem P3, which is created by introducing dual variables \( \lambda \in \mathbb{R}^{n_{\tau}} \) for the equality constraints (23) and dual variables \( \mu \in \mathbb{R}^{n_{\tau}} \) for the inequality constraints (24). It is shown in [2] that the dual problem can be written as:
\[
\max_{\lambda, \mu \geq 0} D(\lambda, \mu)
\]

where \( D(\lambda, \mu) \) is the dual cost function:

\[
D(\lambda, \mu) = -\frac{1}{2} (\overline{A}^T \lambda + \overline{C}^T \mu) \overline{H}^{-1} (\overline{A} \lambda + \overline{C} \mu) - \lambda^T (\overline{B} \lambda - \overline{G}) - \mu^T \overline{d}(1 - \delta)
\]

\[
(33)
\]

In order to perform distributedly the iterations of the dual accelerated gradient method, the following vector \( Y_i \in \mathbb{R}^{n_i} \) of decision variables, associated to the \( i \)-th subsystem, is introduced:

\[
Y_i = [x_{i,1}, u_{i,1}, x_{i,2}, u_{i,2}, \ldots, x_{i,N}]
\]

Also, let \( \lambda_i \in \mathbb{R}^{n_i} \) and \( \mu_i \in \mathbb{R}^{n_i} \) be the dual variables for the equality and the inequality constraints, related to the \( i \)-th subsystem. Then, the distributed iterations of the dual accelerated gradient method are:

\[
Y_i = -\overline{H}_i^{-1} \sum_{j=1}^{M} \overline{A}^T_j \lambda_j + \overline{C}^T_j \mu_j
\]

\[
(35)
\]

\[
\overline{Y}_i = Y_i + \frac{r - 1}{r + 2} (Y_i - Y_i^{-1})
\]

\[
(36)
\]

\[
\lambda_i^{-1} = \lambda_i^0 + \frac{r - 1}{r + 2} (\lambda_i^0 - \lambda_i^{r-1}) + \frac{1}{L} (\overline{A} \lambda_i - (\overline{B} \lambda_i - \overline{G})
\]

\[
(37)
\]

\[
\mu_i^{r+1} = \max(0, \mu_i^0 + \frac{r - 1}{r + 2} (\mu_i^0 - \mu_i^{r-1}) + \frac{1}{L} (\overline{C} \overline{Y}_i - \overline{d}(1 - \delta)) \]

\[
(38)
\]

\[
i = 1, 2, \ldots, M
\]

where \( \overline{A}_j \) are the columns of the matrix \( \overline{A} \) corresponding to the decision vector \( Y_i \). Here, \( L = \| \overline{A}^T \overline{C} \| \overline{H}^{-1} [\overline{A}^T \overline{C}] \) is the Lipschitz constant to the gradient of the dual function (33) and \( r \) is the iteration number. Note that because of the couplings in the dynamics models of the subsystems, in (35) the computation of the decision variables \( Y_i \) for the \( i \)-th subsystem requires to have information about the dual variables \( \lambda_i \) for the whole system. For the same reason, in (37) the update of the dual variables \( \lambda_i \) associated to the \( i \)-th subsystem uses the information about the decision variables \( \overline{Y}_i \) for the entire system. Since there are no couplings in the control input and state constraints of the subsystems (cf. (4)-(6)), in (38) the update of the dual variables \( \mu_i \) for the \( i \)-th subsystem requires information only about the decision variables \( \overline{Y}_i \) for this subsystem.

In [1], an algorithm for distributed quasi-NMPC is developed, which includes two loops. In the outer loop, the dynamics of the nonlinear system (3) is locally approximated with a linear model (14)-(15) about the current guess for the control input trajectory and the corresponding state trajectory of the system (3). Then, in the inner loop, a suboptimal solution to the resulting QP problem P3 is found by applying the distributed iterations (35)-(38) of the dual accelerated gradient method.

**Dual-mode distributed MPC**

Here, a dual-mode distributed MPC strategy is proposed which consists in using the distributed quasi-NMPC approach when the system is far from equilibrium, and the linear distributed MPC method [2] when it is close to the origin. It is assumed that the dynamics of the subsystems in a neighborhood of the origin are described by the following linear discrete-time models:

\[
x_i(t+1) = \sum_{j=1}^{M} (A_{ij}^o x_j(t) + B_{ij}^o u_j(t)), i = 1, \ldots, M
\]

\[
(39)
\]

where the matrices \( A_{ij}^o, B_{ij}^o \) are determined as:

\[
A_{ij}^o = \nabla_x f_i(0,0), B_{ij}^o = \nabla_u f_i(0,0), i, j = 1, \ldots, M
\]

\[
(40)
\]

It is supposed that a full measurement \( \overline{x} = [\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_M] \) of the overall state is available at the current time \( t \) and the state constraints are satisfied in the neighborhood of the origin. Then, for the current overall state \( \overline{x} \), the regulation linear MPC solves the optimization problem:

**Problem P4 (Centralized linear MPC):**

\[
V^{opt}(\overline{x}) = \min_{U} J(U, \overline{x})
\]

subject to \( x_i = \overline{x} \) and:

\[
u_{i,t+k} \in U_i, i = 1, \ldots, M, k = 0, 1, \ldots, N-1
\]

\[
(42)
\]
\[ x_{i,t+k|t} = \sum_{j=1}^{M} \left( A_{ij}^o x_{j,t+k|t} + B_{ij}^o u_{j,t+k} \right) \]
\[ i = 1, \ldots, M, \ \ k = 0, 1, \ldots, N - 1 \]  

where the cost function is given by (13). The problem P4 is represented as the following QP problem:

**Problem P5 (QP problem):**
\[ V^*(\bar{x}) = \min_y \frac{1}{2} y^T \bar{H}^o y \]
subject to:
\[ \bar{A}^o y = \bar{B}^o \bar{x} \]
\[ \bar{C}^o y \leq (1 - \delta) \bar{d}^o \]

where the vector \( y \) is defined by (21) and the matrices \( \bar{H}^o, \bar{A}^o, \bar{B}^o, \bar{C}^o \) and \( \bar{d}^o \) are determined in a way similar to that described in Section 3. The problem P5 is solved with a modified version of the distributed iterations (35)–(38) (note that in (37), \( \bar{G}_i = 0, \ i = 1, \ldots, M \)).

**DUAL-MODE DISTRIBUTED MPC OF A QUADRUPLE-TANK SYSTEM**

Here, the dual-model distributed MPC approach is applied to the regulation of a quadruple-tank system.

**System description**

As an example, the quadruple-tank system in [12] is considered, which is schematically shown in Fig. 2. The objective is to control the level in the lower two tanks with two pumps. The control inputs are \( v_1 \) and \( v_2 \) (input voltages to the pumps) and the outputs are \( y_1 \) and \( y_2 \) (voltages from level measurement devices).

The first-principles model, describing the dynamics of the system, is [12]:
\[ \dot{h}_1 = -\frac{a_1}{A_1} \sqrt{2gh_1} + \frac{a_3}{A_1} \sqrt{2gh_3} + \frac{\gamma_1 k}{A_1} v_1 \]  
\[ \dot{h}_2 = -\frac{a_2}{A_2} \sqrt{2gh_2} + \frac{a_4}{A_2} \sqrt{2gh_4} + \frac{\gamma_2 k}{A_2} v_2 \]  
\[ \dot{h}_3 = -\frac{a_3}{A_3} \sqrt{2gh_3} + \frac{\gamma_2 k}{A_3} v_2 \]  
\[ \dot{h}_4 = -\frac{a_4}{A_4} \sqrt{2gh_4} + \frac{(1 - \gamma_2) k}{A_4} v_1 \]  

In (47)-(50), \( A_i \) is the cross-sectional area of tank \( i \), \( a_i \) is the cross-sectional area of the outlet hole of tank \( i \), \( h_i \) is the water level in tank \( i \) [12]. The voltage applied to pump \( i \) is \( v_i \) and the corresponding flow is \( k_i v_i \). The parameters \( \gamma_1, \gamma_2 \in (0, 1) \) are determined from the positions of the two valves. In the simulation experiments, it is chosen that \( \gamma_1 = 0.7 \), \( \gamma_2 = 0.6 \), which lead to a minimum-phase behavior of the plant [12].

The flow to tank 1 is \( \gamma_1 k_1 v_1 \) and the flow to tank 4 is \( (1 - \gamma_1) k_1 v_1 \). The flows to tanks 2 and 3 are \( \gamma_2 k_2 v_2 \) and \( (1 - \gamma_2) k_2 v_2 \), respectively. The acceleration of gravity is denoted \( g \). The measured level signals are \( y_1 = k_1 h_1 \) and \( y_2 = k_c h_2 \), where \( k_c \) is a constant. The parameter values of the quadruple-tank system are given in [12].

The control objective is to keep the water levels \( h_1 \) and \( h_2 \) at the set-points:
\[ h_1^* = 12.4 \text{ cm}, \ h_2^* = 12.7 \text{ cm} \]  

Fig. 2. Quadruple-tank system [12].
The steady-state values of $h_3^*$, $h_4^*$, $v_1^*$, $v_2^*$, corresponding to these set-points are:

$$h_3^* = 1.6 \, \text{cm}, \quad h_4^* = 1.45 \, \text{cm}, \quad v_1^* = 3.04 \, \text{V}, \quad v_2^* = 2.97 \, \text{V}$$

(52)

The following variables are introduced:

$$x_{1,1} = h_1 - h_1^*, \quad x_{1,2} = h_4 - h_4^*$$

$$x_{2,1} = h_2 - h_2^*, \quad x_{2,2} = h_3 - h_3^*$$

(53)

$$u_i = v_i - v_i^*, \quad i = 1, 2$$

(54)

Then, the quadruple-tank system can be considered as consisting of two interconnected sub-systems, which are described by:

Subsystem $S_1$:

$$\dot{x}_{1,1} = -\frac{a_1}{A_1} \sqrt{2g(x_{1,1} + h_1^*)} + \frac{a_1}{A_1} \sqrt{2g(x_{2,2} + h_2^*)} + \frac{\gamma_1 k_1}{A_1} (u_1 + v_1^*)$$

(55)

$$\dot{x}_{1,2} = -\frac{a_4}{A_1} \sqrt{2g(x_{1,2} + h_4^*)} + \frac{(1-\gamma_1)k_1}{A_4} (u_1 + v_1^*)$$

Subsystem $S_2$:

$$\dot{x}_{2,1} = -\frac{a_4}{A_2} \sqrt{2g(x_{2,1} + h_2^*)} + \frac{a_4}{A_2} \sqrt{2g(x_{1,1} + h_1^*)} + \frac{\gamma_2 k_2}{A_2} (u_2 + v_2^*)$$

(57)

$$\dot{x}_{2,2} = -\frac{a_4}{A_3} \sqrt{2g(x_{2,2} + h_2^*)} + \frac{(1-\gamma_2)k_2}{A_4} (u_2 + v_2^*)$$

(58)

The subsystem $S_1$ influences the dynamics of the subsystem $S_2$ with the expression $\frac{a_4}{A_2} \sqrt{2g(x_{1,1} + h_1^*)}$ while the subsystem $S_2$ influences the dynamics of the subsystem $S_1$ with the expression $\frac{a_4}{A_3} \sqrt{2g(x_{2,2} + h_2^*)}$.

**Simulation results**

The performance of the proposed dual-mode distributed MPC approach is studied by simulations for the quadruple-tank system described above. The ordinary differential equations (55)-(58) are discretized with sampling time of 1 s by applying the Euler’s method with step 0.1 s. The constraints imposed on the system (47)-(50) are:

$$0 \leq v_i(t) \leq 6 \, \text{V}, \quad i = 1, 2$$

(59)

$$0 \leq h_i(t) \leq 20 \, \text{cm}, \quad i = 1, 2$$

(60)

$$0 \leq h_i(t) \leq 3 \, \text{cm}, \quad i = 3, 4$$

which by taking into account (51)-(54) become:

$$-3.04 \leq u_i(t) \leq 2.96 \, \text{V}$$

(61)

$$-2.97 \leq u_2(t) \leq 3.03 \, \text{V}$$

$$-12.4 \leq x_{1,1}(t) \leq 7.6 \, \text{cm}$$

$$-1.45 \leq x_{1,2}(t) \leq 1.55 \, \text{cm}$$

$$-12.7 \leq x_{2,1}(t) \leq 7.3 \, \text{cm}$$

$$-1.60 \leq x_{2,2}(t) \leq 1.40 \, \text{cm}$$

(62)

(63)

The prediction horizon in the centralized MPC problem is $N = 10$ and the weighting matrices in the cost function (13) are $Q = Q_2 = \text{diag}(10, 1)$, $R_1 = R_2 = 0.1$. The distributed MPC approach is used to generate the two control inputs for initial states of the subsystems $S_1$ and $S_2$:

$$[x_{1,1}(0) \ x_{1,2}(0) \ x_{1,1}(0) \ x_{1,2}(0)] = [-2.4 - 0.45 - 2.7 - 0.60]$$

(64)

The trajectories of the control inputs and the states associated to the two subsystems are depicted in Fig. 3 to Fig. 5.

The trajectories obtained with the suboptimal dual-mode distributed MPC approach are compared to those corresponding to the centralized NMPC approach, which solves problem P1 at each time instant. It can be seen that the distributed suboptimal MPC approach leads to feasible trajectories and the level of suboptimality is acceptable.

**CONCLUSIONS**

In this paper, a dual-mode distributed MPC approach is proposed in order to reduce the on-line computational complexity of the distributed optimal control of nonlinear interconnected systems. It consists in using a quasi-nonlinear distributed MPC approach when the state variables are far from the origin and a linear
Fig. 3. The control inputs for the two subsystems.

Fig. 4. The states of subsystem $S_1$.

Fig. 5. The states of subsystem $S_2$. 
distributed MPC method in a neighborhood of the origin. The proposed method is illustrated with simulations on the model of a quadruple-tank system.

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